

SUSY N -supergroups and their real forms

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Abstract

We study SUSY N -supergroups, $N = 1, 2$, their classification and explicit realization, together with their real forms. In the end, we give the supergroup of SUSY preserving automorphism of $\mathbb{C}^{1|1}$ and we identify it with a subsupergroup of the SUSY preserving automorphisms of $\mathbb{P}^{1|1}$.

1 Introduction

The papers [5] and [6] carry out a thorough study of the real compact supergroups $S^{1|1}$ and $S^{1|2}$, called *supercircles* in odd dimension 1 and 2, and their theory of representation, together with the Peter-Weyl theorem. These supercircles are realized as real forms of $(\mathbb{C}^{1|1})^\times$ and $(\mathbb{C}^{1|2})^\times$ respectively, and in the case of $S^{1|1}$, we have a precise relation between the real structures and real forms of $(\mathbb{C}^{1|1})^\times$ and the SUSY preserving automorphism of the SUSY 1-curve $(\mathbb{C}^{1|1})^\times$. In this paper we want to study the SUSY N -curves, which also admit a supergroup structure, namely the SUSY N -supergroups.

Our paper is organized as follows.

In Section 2 we give the definition of SUSY 2-supergroup and we classify them. We also give an interpretation of the supergroup $\mathrm{SL}(1|1)$ as the SUSY 2-curve incidence supermanifold of the SUSY 1-curve $(\mathbb{C}^{1|1})^\times$ and its dual. In Section 3 we study of real forms of SUSY N -supergroups and classify them. Finally in Section 4 we compute the supergroup of SUSY preserving automorphism of $\mathbb{C}^{1|1}$.

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2 The SUSY N -supergroups

We want to study SUSY N -curves which also have a supergroup structure. In the following we shall use the notation and terminology as in [18] Ch. 2. X is a *SUSY 1-curve*, if X is a $1|1$ complex supermanifold and there is a $0|1$ distribution \mathcal{D} , such that $\mathcal{D} \otimes \mathcal{D} \cong TX/[\mathcal{D}, \mathcal{D}]$. X is a *SUSY 2-curve*, if it is a $2|1$ complex supermanifold and there are two a $0|1$ distributions \mathcal{D}_i , such $[\mathcal{D}_i, \mathcal{D}_i] \subset \mathcal{D}_i$ and $\mathcal{D}_1 \otimes \mathcal{D}_2 \cong TX/[\mathcal{D}_1, \mathcal{D}_2]$. In [18] Ch. 2, Manin provides local models for such distributions:

$$\mathcal{D} = \zeta \partial_z + \partial_\zeta \quad \text{on } X \text{ a SUSY 1-curve}$$

$$\mathcal{D}_1 = \zeta_2 \partial_z + \partial_{\zeta_1} \quad \mathcal{D}_2 = \zeta_1 \partial_z + \partial_{\zeta_2} \quad \text{on } X \text{ a SUSY 2-curve}$$

Definition 2.1. Let X be a SUSY N -curve, with $0|1$ distribution(s) \mathcal{D}_i , where $i = 1$ for $N = 1$ and $i = 1, 2$ for $N = 2$. We say that X is a *SUSY N -supergroup* if X is a supergroup and the distribution(s) \mathcal{D}_i are left invariant.

We want to classify the SUSY N -supergroups and relate them to Manin's approach to SUSY curves. We shall use interchangeably the formalisms of functor of points and also of super Harish-Chandra pairs (SHCP), namely we describe a supergroup by a pair (G_0, \mathfrak{g}) , where G_0 is a complex group and \mathfrak{g} is a Lie superalgebra (see [3] Ch. 3, 7).

Proposition 2.2. *Up to isomorphism, for $N = 1, 2$ fixed, we have only two SUSY N -supergroups.*

1. For $N = 1$ they are $(\mathbb{C}^{1|1})^\times$ and $\mathbb{C}^{1|1}$ with group law respectively:

$$\begin{aligned} (w, \eta) \cdot (w', \eta') &= (ww' + \eta\eta', w\eta' + \eta w') \\ (z, \zeta) \cdot (z', \zeta') &= (z + z' + \zeta\zeta', \zeta + \zeta') \end{aligned} \tag{1}$$

2. For $N = 2$ they are $(\mathbb{C}^{1|2})^\times$ and $\mathbb{C}^{1|2}$ with group laws:

$$\begin{aligned}
(v, \xi, \eta) \cdot (v', \xi', \eta') &= (vv' + \eta\xi', v\xi' + \xi v' + \xi v^{-1}\eta\xi', \\
&\quad \eta v' + v\eta' + \eta\xi'v'^{-1}\eta')
\end{aligned} \tag{2}$$

$$(z, \zeta, \chi) \cdot (z', \zeta', \chi') = (z + z' + \zeta\chi', \zeta + \zeta', \chi + \chi')$$

Proof. For $N = 1$ the statements are contained in [5], provided that one verifies left invariance, which is a straightforward check. Let $N = 2$, \mathcal{D}_i the left invariant distributions on the SUSY 2-supergroup X . Let $D_i \in \text{Lie}(X)$ be a left invariant (odd) generator of \mathcal{D}_i . We have $\text{Lie}(X) = \langle D_1, D_2, [D_1, D_2] \rangle$. Notice that in general $[\mathcal{D}_i, \mathcal{D}_i] \neq 0$, however since $D_i \in \text{Lie}(X)$ and the bracket must preserve the parity, we have $[D_i, D_i] = 0$. Let $C := [D_1, D_2]$. The Jacobi identity gives immediately that C is central and then an easy calculation shows that the given group laws correspond to the Lie superalgebra we have obtained respectively for $G_0 = \mathbb{C}^\times$ and $G_0 = \mathbb{C}$. \square

Remark 2.3. We can interpret the multiplicative and additive SUSY 2-supergroups using matrix supergroups. $(\mathbb{C}^{1|2})^\times$ is $\text{SL}(1|1)$. In functor of points notation:

$$(\mathbb{C}^{1|2})^\times(T) = \text{SL}(1|1)(T) = \left\{ \begin{pmatrix} u & \xi \\ \eta & v \end{pmatrix} \mid v^{-1}(u - \xi v^{-1}\eta) = 1 \right\}$$

$\mathbb{C}^{1|2}$ is the subgroup of $\text{SL}(2|1)$ given in the functor of points notation by:

$$\mathbb{C}^{1|2}(T) = \left\{ \begin{pmatrix} 1 & z & \zeta \\ 0 & 1 & 0 \\ 0 & \chi & 1 \end{pmatrix} \right\}$$

This approach can be helpful in calculations. We leave to the reader the straightforward checks regarding the group laws.

We now want to interpret some of the discussion in [18] Sec. 6 in the framework of SUSY supergroups. Let X be a SUSY 1-curve and \widehat{X} its dual. The T points of \widehat{X} are the $0|1$ subvarieties of $X(T)$. Let Δ be the superdiagonal subscheme of $X \times \widehat{X}$. It is locally defined by the incidence relation:

$$z - z' - \zeta'\zeta = 0 \tag{3}$$

(z, ζ') and (z', ζ') local coordinates of X and \widehat{X} (see Def. 6.2 in [18]). Δ is a SUSY 2-curve, with distributions $\mathcal{D}_1, \mathcal{D}_2$ and we have the commutative diagram:

$$\begin{array}{ccc} & \Delta & \\ \swarrow & & \searrow \\ X = \Delta/\mathcal{D}_1 & \longrightarrow & \Delta/\mathcal{D}_2 = \widehat{X} \end{array} \quad (4)$$

where Δ/\mathcal{D}_i means that we are considering the subsheaf of \mathcal{O}_Δ consisting of sections which are invariant under \mathcal{D}_i . This diagram gives the natural isomorphism $X \cong \widehat{X}$.

On $(\mathbb{C}^{1|2})^\times$ we have the global SUSY 2-structure:

$$D_1 = \partial_{\zeta_1} + \zeta_2 \partial_z \quad D_2 = \partial_{\zeta_2} + \zeta_1 \partial_z \quad (5)$$

Clearly $D_1^2 = D_2^2 = 0$ and $[D_1, D_2] = 2\partial_z$. In Remark 2.3, we have viewed $(\mathbb{C}^{1|2})^\times$ as the supergroup $\mathrm{SL}(1|1)$. The condition on the berezinian is the (global) incidence relation (3) and allow us to identify $\mathrm{SL}(1|1)$ with Δ for $X = (\mathbb{C}^{1|1})^\times$ and $\mathrm{GL}(1|1)$ with $X \times \widehat{X}$, Notice that $X, \widehat{X} \subset \mathrm{SL}(1|1)$ as the sub-supergroups:

$$X(T) = \left\{ \begin{pmatrix} x & \xi \\ \xi & x \end{pmatrix} \right\}, \quad \widehat{X}(T) = \left\{ \begin{pmatrix} y & \eta \\ -\eta & y \end{pmatrix} \right\}$$

These inclusions correspond to the Lie superalgebras inclusions:

$$\langle C, U = E + F \rangle, \quad \langle C, V = E - F \rangle \subset \langle C, E, F \rangle = \mathfrak{sl}(1|1)$$

where C, E, F are the usual generators for $\mathfrak{sl}(1|1)$, namely:

$$[C, E] = [C, F] = [E, E] = [F, F] = 0, \quad [E, F] = C \quad (6)$$

It is important to remark that while the X and \widehat{X} embed into $\Delta = \mathrm{SL}(1|1)$ as its sub-supergroups, the arrows in (4) are not supergroup morphisms.

3 Real forms of SUSY supergroups

We want to study the real forms of SUSY N -supergroups. In [5] and [6] we proved that, up to isomorphism, there is one real form of the SUSY 1-supergroup $(\mathbb{C}^{1|1})^\times$ and the corresponding involution is the composition of complex conjugation and the SUSY preserving automorphisms P_\pm . We wish to prove a similar result for the SUSY 2-supergroups.

Definition 3.1. Let X be a SUSY 2-curve with distributions \mathcal{D}_i . We say that an automorphism $\phi : X \rightarrow X$ is *SUSY preserving* if $\phi_*(\mathcal{D}_i) = \mathcal{D}_j$, that is, if ϕ preserves individually each distribution or exchanges them. If X is a SUSY 2-supergroup we furtherly ask ϕ to be a supergroup automorphism.

Notice that in a SUSY 2 curve the roles of \mathcal{D}_1 and \mathcal{D}_2 are interchangeable; this forces us to give such a definition of SUSY preserving automorphism.

We start our discussion by observing that, up to isomorphism, there is only one real form of the Lie superalgebra $\mathfrak{sl}(1|1)$ with compact even part. In fact, assume $\mathfrak{g}_{\mathbb{R}} = \text{span}_{\mathbb{R}}\{iC, U, V\}$ is such real form, with central even element iC (see (6)). If $U = aE + bF$, $V = cE + dF$, there is no loss of generality in assuming $a = 1$ because $E \mapsto a^{-1}E$, $F \mapsto aF$, $C \mapsto C$ is a Lie superalgebra automorphism of $\mathfrak{sl}(1|1)$. Assume furtherly $[U, U] \neq 0$ (when both $[U, U] = [V, V] = 0$ we leave to the reader the easy check of what the real form is). Then we have that $b = i$, up to a constant, that we absorbe in C . An easy calculation shows that $V = iE - F$, hence we have proven the following proposition.

Proposition 3.2. *Up to isomorphism, there is a unique real form of $\mathfrak{sl}(1|1)$ with compact even part, namely*

$$\mathfrak{su}(1|1) = \text{span}_{\mathbb{R}}\{iC, U = E + iF, V = iE - F\} \subset \mathfrak{sl}(1|1).$$

We can then state the theorem giving all real forms of SUSY 2-supergroups with compact support.

Theorem 3.3. *Up to isomorphism, there exists a unique real form of the SUSY 2-supergroup $\text{SL}(1|1)$ and it is obtained with an involution $\sigma = c \circ \phi$ where ϕ is a SUSY preserving automorphism and c is a complex conjugation. Explicitly:*

$$\sigma \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} = \begin{pmatrix} \bar{d}^{-1} & -i\bar{a}^{-2}\bar{\gamma} \\ -i\bar{a}^{-2}\bar{\beta} & \bar{a}^{-1} \end{pmatrix}$$

Proof. We first check that the given σ is of the prescribed type, namely that

$$\phi \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} = \begin{pmatrix} d^{-1} & -ia^{-2}\gamma \\ -ia^{-2}\beta & a^{-1} \end{pmatrix}$$

is a SUSY preserving automorphism. The fact it is a supergroup automorphism is a simple check. The SUSY preserving property can be verified at the Lie superalgebra level and we leave it to the reader.

As for uniqueness, by Prop. 3.2 we know there exists a unique real form $\mathfrak{su}(1|1)$ of $\mathfrak{sl}(1|1) = \text{Lie}(\text{SL}(1|1))$ and one readily checks $\mathfrak{su}(1|1) = \text{Lie}(\text{SU}(1|1))$. By the equivalence of categories in SHCP theory we obtain the result. \square

4 The SUSY preserving automorphisms of $\mathbb{C}^{1|1}$

Let our notation and terminology be as in [5], [6].

On $\mathbb{C}^{1|1}$ we have the globally defined SUSY structure given by the vector field:

$$D = \partial_\zeta + \zeta \partial_z$$

where (z, ζ) are the global coordinates. This structure is unique up to isomorphism (see [13] Sec. 4). We want to determine the supergroup of automorphism of $\mathbb{C}^{1|1}$ preserving such SUSY structure. We will denote it with $\text{Aut}_{\text{SUSY}}(\mathbb{C}^{1|1})$. In the work [13] we have provided the \mathbb{C} -points of such supergroup; they are obtained by looking at the transformations leaving invariant the 1-form:

$$s = dz - \zeta d\zeta$$

and are given by the endomorphisms

$$F(z, \zeta) = (az + b, \sqrt{a}\zeta)$$

We can identify the \mathbb{C} -points of the supergroup of SUSY preserving automorphism with the matrix group

$$\text{Aut}_{\text{SUSY}}(\mathbb{C}^{1|1})(\mathbb{C}) = \left\{ \begin{pmatrix} c & d & 0 \\ 0 & c^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid a, b \in \mathbb{C} \right\} \subset \text{Aut}_{\text{SUSY}}(\mathbb{P}^{1|1})(\mathbb{C}) \quad (7)$$

This is a subgroup of the \mathbb{C} -points of the SUSY-preserving automorphisms of the SUSY 1-curve $\mathbb{P}^{1|1}$, namely those fixing the point at infinity (see Sec. 5 [13] and Sec. 5). In such identification $a = c^2$ and $b = dc$. Notice that, though $\text{Aut}_{\text{SUSY}}(\mathbb{C}^{1|1})(\mathbb{C})$ is a matrix group, it is not obvious that also $\text{Aut}_{\text{SUSY}}(\mathbb{C}^{1|1})$ should be, since we are looking at the SUSY preserving automorphism of $\mathbb{C}^{1|1}$ as supermanifold morphisms. Nevertheless we will show that this is the case.

An automorphism $F : \mathbb{C}^{1|1} \longrightarrow \mathbb{C}^{1|1}$ induces an automorphism $F^* : \mathcal{O}(\mathbb{C}^{1|1}) \longrightarrow \mathcal{O}(\mathbb{C}^{1|1})$ of the superalgebra of global sections. F is SUSY preserving if and only if

$$F^* \circ D = kD \circ F^* \quad (8)$$

where D is now interpreted as a derivation of $\mathcal{O}(\mathbb{C}^{1|1})$ and k is a suitable constant. We first consider the infinitesimal picture and compute $\text{Lie}(\text{Aut}_{\text{SUSY}}(\mathbb{C}^{1|1}))$. By (7), $\text{Lie}(\text{Aut}_{\text{SUSY}}(\mathbb{C}^{1|1}))_0$ is 2 dimensional, and as one can readily check, it is spanned by the two even vector fields:

$$U_1 = 2z\partial_z + \zeta\partial_\zeta, \quad U_2 = \partial_z$$

We hence only need to compute $\text{Lie}(\text{Aut}_{\text{SUSY}}(\mathbb{C}^{1|1}))_1$.

Proposition 4.1. *$\text{Lie}(\text{Aut}_{\text{SUSY}}(\mathbb{C}^{1|1}))$ is the Lie subsuperalgebra of the vector fields on $\mathbb{C}^{1|1}$ spanned by*

$$U_1 = 2z\partial_z + \zeta\partial_\zeta, \quad U_2 = \partial_z, \quad V = \zeta\partial_z - \partial_\zeta.$$

with brackets:

$$[V, V] = 2U_2, \quad [U_2, U_1] = -2U_1, \quad [U_2, V] = -V, \quad [U_1, V] = 0$$

Proof. Consider $I + \theta\chi$, for $\chi \in \text{Lie}(\text{Aut}_{\text{SUSY}}(\mathbb{C}^{1|1}))_1$. The condition (8) gives immediately that the odd derivation χ^* of $\mathcal{O}(\mathbb{C}^{1|1})$ induced by χ must satisfy $[\chi^*, D] = 0$. A small calculation gives then the result. \square

In the Super Harish-Chandra pair (SHCP) formalism, we can immediately write the supergroup of SUSY preserving automorphism:

$$\text{Aut}_{\text{SUSY}}(\mathbb{C}^{1|1}) = (\text{Aut}_{\text{SUSY}}(\mathbb{C}^{1|1})(\mathbb{C}), \text{Lie}(\text{Aut}_{\text{SUSY}}(\mathbb{C}^{1|1})))$$

The next proposition identifies such supergroup with a natural subsupergroup of $\text{Aut}_{\text{SUSY}}(\mathbb{P}^{1|1}) = \text{SpO}(2|1)$ (Ref. [14]) using the more geometric functor of points approach.

Proposition 4.2. *$\text{Aut}_{\text{SUSY}}(\mathbb{C}^{1|1})$ is the stabilizer subsupergroup in $\text{SpO}(2|1)$ of the point at infinity:*

$$\text{Aut}_{\text{SUSY}}(\mathbb{C}^{1|1})(T) = \left\{ \begin{pmatrix} c & d & \gamma \\ 0 & c^{-1} & 0 \\ 0 & c^{-1}\gamma & 1 \end{pmatrix} \mid c, d \in \mathcal{O}(T)_0, \gamma \in \mathcal{O}(T)_1 \right\}$$

$T \in (\text{smflds})_{\mathbb{R}}$.

Proof. The first statement is an immediate consequence of Proposition 4.1. As for the second one, consider the subgroup G of $\mathrm{SpO}(2|1) = \mathrm{Aut}_{\mathrm{SUSY}}(\mathbb{P}^{1|1})$ that fixes the point at infinity. Its functor of points is given by:

$$G(T) = \left\{ \begin{pmatrix} c & d & \gamma \\ 0 & c^{-1} & 0 \\ 0 & c^{-1}\gamma & 1 \end{pmatrix} \mid c, d \in \mathcal{O}(T)_0, \gamma \in \mathcal{O}(T)_1 \right\}$$

G is representable and its SHCP coincides with $\mathrm{Aut}_{\mathrm{SUSY}}(\mathbb{C}^{1|1})$, because $\mathrm{Lie}(G) = \mathrm{Lie}(\mathrm{Aut}_{\mathrm{SUSY}}(\mathbb{C}^{1|1}))$, $|G| = \mathrm{Aut}_{\mathrm{SUSY}}(\mathbb{C}^{1|1})(\mathbb{C})$ and we have the compatibility conditions. \square

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